Math 246B Lecture 17 Notes

Daniel Raban

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1 Factorization of Entire Functions of Finite Order

1.1 Number of zeros of entire functions of finite order

Last time, we proved Jensen's formula.

Theorem 1.1. *Let f be entire of finite order ρ, and let $n(r) = |\{z : |z| < r, f(z) = 0\}|$. Then for all $\varepsilon > 0$ there exists a constant C_ε such that*

$$n(r) \leq C_\varepsilon r^{\rho+\varepsilon}$$

for all $r \geq 1$.

Proof. If $f(0) \neq 0$, then

$$\int_0^{2r} \frac{n(t)}{t}\, dt \geq \int_r^{2r} \frac{n(t)}{t}\, dt = n(r)\log(2),$$

where the inequality comes from the fact that n is increasing. Using Jensen's formula,

$$\log(2)n(r) \leq \frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\varphi})|\, d\varphi + C \leq C_\varepsilon + Cr^{\rho+\varepsilon} + C \leq C_\varepsilon r^{\rho+\varepsilon}.$$

If $f(0) = 0$, apply the previous argument to $g(z) = f(z)/z^m$, where m is the multiplicity of 0. Since $n(r) = n_g(r) + m$, we get the result. $\square$

1.2 Weierstrass factors and Weierstrass' theorem for $\mathbb{C}$

Definition 1.1. When $m \geq 0$ is an integer, we define the **Weierstrass factors**[1] as

$$E_m(z) = (1-z)e^{\sum_{i=1}^m z^j/j}.$$

[1] Weierstrass used these in his proof of Weierstrass' theorem. We did not.

Remark 1.1. We would like to consider infinite products of the form

$$\prod(1 - z/a_k)e^{-g(z/a_k)},$$

where $|a_k| \to \infty$ and where g should approximate $\log(1-z) = -\sum_{j=1}^{\infty} z^j/j$ for $|z| < 1$. The idea of the Weierstrass factors is that the factors are the partial sums of this approximation.

Lemma 1.1. *For all* $|z| < 1$*,*

$$|1 - E_m(z)| \leq |z|^{m+1}.$$

Proof. Let $h(z) = 1 - E_m(z)$, so $h(0) = 0$. Compute

$$h'(z) = e^{\sum_{j=1}^{m} z^j/j}(1 + z\varphi'(z) - \varphi'(z))' = z^m e^{\sum_{j=1}^{m} z^j/j}.$$

So $h(z) = O(|z|^{m+1})$, and we see that $h(z)/z^{m+1}$ is holomorphic on $\mathbb{C}$. We have

$$h'(z) = z^m(1 + a_1 z + a_2 z^2 + \cdots)$$

with $a_j \geq 0$ for all j. Integrating, we get

$$h(z) = z^{m+1}(b_0 + b_1 z + b_2 z^2 + \cdots),$$

with $b_j \geq 0$ for all j. If we write $g(z) = h(z)/z^{m+1}$, then

$$|g(z)| \leq g(|z|) \leq g(1) = h(1) = 1.$$

□

Theorem 1.2 (Weierstrass' theorem for $\mathbb{C}$)**.** *Let* $(a_k)_{k=1}^{\infty}$ *be a sequence in* $\mathbb{C} \setminus \{0\}$ *such that* $|a_k| \to \infty$ *as* $k \to \infty$*. Then the canonical product*

$$f(z) = \prod_{k=1}^{\infty} E_k(z/a_k)$$

converges locally uniformly in $\mathbb{C}$ *and defines an entire function* f *such that* $f^{-1}(\{0\}) = \{a_k\}$ *and the multiplicity of* $a \in f^{-1}(\{0\})$ *is the number of* k *such that* $a = a_k$*.*

Proof. It suffices to check that for any compact set $K \subseteq \mathbb{C}$,

$$\sum_{k=1}^{\infty} \sup_K |1 - E_k(z/a_k)| < \infty.$$

$K \subseteq \{|z| \leq |a_k|/2\}$ for all k large enough, and by the lemma,

$$|a - E_k(z/a_k)| \leq |z/a_k|^{k+1} \leq 2^{-k}.$$

The result follows. □

1.3 Factorization of entire functions of finite order

Now assume that f is entire of finite order ρ with the zeros $a_k \neq 0$ counted with multiplicities such that $|a_1| \leq |a_2| \leq \cdots$ and $|a_k| \to \infty$.

Proposition 1.1. *The series*

$$\sum_{k=1}^{\infty} \frac{1}{|a_k|^{m+1}} < \infty.$$

provided that $m > \rho - 1$.

Proof. Write

$$\begin{aligned}\sum_{|a_k|\geq 1} |a_k|^{-m-1} &= \sum_{j=0}^{\infty} \underbrace{\left(\sum_{2^j \leq |a_k| \leq a^{j+1}} |a_k|^{-m-1} \right)}_{2^{-j(m+1)} n(2^{j+1})} \\ &\leq \sum_{j=0}^{\infty} C_\varepsilon 2^{(j+1)(\rho+\varepsilon)} 2^{-j(m+1)} \\ &\leq C_\varepsilon \sum_{j=0}^{\infty} 2^{j(\rho+\varepsilon-m-1} < \infty\end{aligned}$$

if $\rho + \varepsilon < m + 1$. $\square$

Proposition 1.2. *Let m be the smallest integer such that $m > \rho - 1$ (so that $m \leq \rho < m + 1$). The canonical product*

$$\prod_{k=1}^{\infty} E_m(z/a_k)$$

converges locally uniformly in $\mathbb{C}$.

Remark 1.2. The improvement here is that we can use a fixed Weierstrass factor here instead of having it depend on k.

Proof. If $|z| < a_k/2$, then $|1 - E_m(z/a_k)| \leq |z/a_k|^{m+1}$. So for compact $K \subseteq \mathbb{C}$,

$$\sum \sup_K |1 - E_m(z/a_k)| < \infty. \qquad \square$$

To summarize, we can write:

$$f(z) = e^{g(z)} z^p \prod_{k=1}^{\infty} E_m(z/a_k),$$

where o is the multiplicity of 0 as the zero of f, and g is entire. This will allow us to understand the structure of entire functions of finite order in the following way:

Theorem 1.3 (Hadamard)**.** *The function g is a polynomial of degree $\leq \rho$.*